

# Relativistic two-fluid hydrodynamics with quantized vorticity from the nonlinear Klein-Gordon equation

Chi Xiong<sup>1,\*</sup> and Kerson Huang<sup>2,†</sup>

<sup>1</sup>*Institute of Advanced Studies & School of Physical and Mathematical Science,  
Nanyang Technological University, 639673 Singapore*

<sup>2</sup>*Physics Department, Massachusetts Institute of Technology, Cambridge, MA, USA 02139*

## Abstract

We consider a relativistic two-fluid model of superfluidity, in which the superfluid is described by an order parameter that is a complex scalar field satisfying the nonlinear Klein-Gordon equation (NLKG). The coupling to the normal fluid is introduced via a covariant current-current interaction, which results in the addition of an effective potential, whose imaginary part describes particle transfer between superfluid and normal fluid. Quantized vorticity arises in a class of singular solutions and the related vortex dynamics is incorporated in the modified NLKG, facilitating numerical analysis which is usually very complicated in the phenomenology of vortex filaments. The dual transformation to a string theory description (Kalb-Ramond) of quantum vorticity, the Magnus force and the mutual friction between quantized vortices and normal fluid are also studied.

Keywords: relativistic superfluidity, nonlinear Klein-Gordon field theory, quantized vortices, two-fluid model, Kalb-Ramond field, global string.

PACS number: 11.10.Lm, 11.25.Sq, 03.75.Kk, 03.75.Lm, 67.25.dm

---

\*xiongchi@ntu.edu.sg

†kerson@mit.edu. Deceased 1 September 2016.

## I. INTRODUCTION

Superfluidity is a macroscopic manifestation of quantum phase coherence, and can be described in terms of a complex order parameter, which in the relativistic domain satisfies a nonlinear Klein-Gordon equation (NLKG). We regard the order parameter as the basic variable describing the superfluid, and hydrodynamic variables, such as the superfluid density and velocity, as derived quantities. Such a treatment not only conveys a more accurate physical picture, but is also a highly efficient way to do numerical computations. In a previous paper [1], we consider a pure superfluid at absolute zero. In this paper we extend the discussion to finite temperatures, where there is also a normal fluid.

Being a manifestation of quantum phase coherence over macroscopic distances, superfluidity is best described in terms of a complex order parameter, which in the non-relativistic regime corresponds to a wave function satisfying the nonlinear Schrödinger equation (NLSE). In the relativistic domain this is generalized to the NLKG. We regard the order parameter as the primary state variable of a superfluid, while hydrodynamic quantities, such as the superfluid density and velocity, are derived quantities. Such a view not only gives a more concrete physical picture, but, as shown in [1], also facilitates numerical analysis, especially in regard to quantized vorticity. We shall introduce the normal fluid in the same framework, and, to put it in historical perspective, start with a brief review of the two-fluid model [2, 3].

Shortly after superfluidity was discovered in liquid  $^4\text{He}$  below the critical temperature of 2.8 K [4], as an apparent absence of viscosity, Tisza [2] suggested that the liquid be modeled as two inter-penetrating liquids, one having “super” qualities, and the other behaving in a “normal” fashion. Landau [3] made the model more concrete by regarding the superfluid as the ground state of a quantum mechanical many-body system, and the normal fluid as a system of quasiparticle excitations. The mass density  $\rho$  and mass current density  $\mathbf{j}$  are split into superfluid and normal fluid contributions. In the non-relativistic regime one writes

$$\begin{aligned}\rho &= \rho_s + \rho_n, \\ \mathbf{j} &= \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n,\end{aligned}\tag{1}$$

where the subscripts  $s$  and  $n$  refer respectively to super and normal fluid, with the condition

$$\nabla \times \mathbf{v}_s = 0.\tag{2}$$

The non-relativistic two-fluid hydrodynamics consists of phenomenological equations based on conservation and thermodynamic laws [3][5]:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} &= 0, \\ \left( \frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla \right) \mathbf{v}_s &= -\nabla \mu, \\ \frac{\partial j^k}{\partial t} + \partial_j \Pi^{jk} &= 0, \\ \frac{\partial S}{\partial t} + \nabla \cdot (S \mathbf{v}_n) &= 0,\end{aligned}\tag{3}$$

where  $\mu$  is the chemical potential,  $S$  the entropy density, and  $\Pi^{jk}$  is the energy-momentum tensor:

$$\Pi^{jk} = \rho_s v_s^j v_s^k + \rho_n v_n^j v_n^k + p \delta_{jk},\tag{4}$$

where  $p$  is the pressure. The second equation in (3) is the analog of the Euler equation. At absolute zero  $S$  vanishes identically, and the equation for  $j^k$  becomes the same as the Euler equation, and (3) collapses to the first two equations describing a pure superfluid.

As thermodynamic functions, the quantities  $S, \mu, p$  are to be specified in a more detailed model. They can be calculated, for example, if the normal fluid is modeled at low temperatures as a dilute gas of quasiparticles. Hill and Roberts [6] introduced a special pressure term in  $\mu$ , in order to describe the healing length, the characteristic distance within which the superfluid density decreases to zero at a wall. Geurst [7] has given a general formulation of the two-fluid hydrodynamics in term of an action principle, and a historical review. A relativistic action principle is discussed by Lebedev and Khalatnikov [8].

Even with a built-in healing length, however, the hydrodynamic equations fail to describe one of the signature properties of a superfluid, namely, quantized vorticity. In fact, the irrotational condition (2) rules out vorticity, and to accommodate that one has add it “by hand”, by writing something like  $\mathbf{v}_s = \nabla\alpha + \mathbf{b}$ , where  $\nabla \times \mathbf{b} \neq 0$ , and go through another round of phenomenology for  $\mathbf{b}$ . But all this still does not explain why the vorticity should be quantized, not to mention the impracticality of numerical analysis.

All these difficulties in describing the superfluid are resolved by using a complex order parameter

$$\Psi(\mathbf{r}, t) = F(\mathbf{r}, t) e^{i\beta(\mathbf{r}, t)}, \quad (5)$$

a non-relativistic wave function satisfying a nonlinear Schrödinger equation (NLSE). The superfluid velocity is related to the phase of the wave function through

$$\mathbf{v}_s = \frac{\hbar}{m} \nabla\beta, \quad (6)$$

where  $m$  is the mass scale in the NLSE. The healing length arises automatically, since the superfluid density  $\rho_s = mF^2$  goes to zero continuously at a boundary. The Hill-Roberts pressure is just the “quantum pressure” arising naturally from the NLSE. The quantization of vorticity, namely

$$\oint_C d\mathbf{s} \cdot \mathbf{v}_s = \frac{2\pi\hbar}{m} n, \quad (n = 0, \pm 1, \pm 2, \dots), \quad (7)$$

where  $C$  is a closed contour in space, follows from the fact that the phase  $\beta$  must be a continuous function [10]. In general  $\nabla \times \mathbf{v}_s \neq 0$ , even though  $\mathbf{v}_s$  is a gradient, because  $\Psi$  can develop zeros, thus rendering the space non-simply connected, i.e, admitting closed contours that cannot be deformed to zero continuously. Another advantage of the NLSE is that it can be handled numerically very efficiently.

Adopting the NLSE means that we regard the complex wave function  $\Psi$  as the fundamental variable, and the hydrodynamic quantities  $\rho_s, \mathbf{v}_s$  as derived ones. Thus, the first two equations in (3) are replaced by and implied by the NLSE.

To include the normal fluid in the NLSE, we need to introduce four new degrees of freedom associated with  $\rho_n, \mathbf{v}_n$ . Bogolubov apparently was the first to suggest, in an unpublished note [14], the introduction of gauge-like potentials  $\varphi, \mathbf{A}$  via the transformation

$$\begin{aligned} \frac{\partial}{\partial t} &\rightarrow \nabla - i\varphi \\ \nabla &\rightarrow \nabla - i\mathbf{A} \end{aligned} \quad (8)$$

This is done in order to couple the new degrees of freedom to the phase of the wave function; the system is of course not locally gauge-invariant. (It had better not be, for otherwise the above would have no physical effect.) Coste [15] shows how one can relate  $\varphi$ ,  $\mathbf{A}$  to  $\rho_n$ ,  $\mathbf{v}_n$  through considerations based on Galilean invariance. To obtain the equations of motion for  $\rho_n$ ,  $\mathbf{v}_n$ , Coste uses a hybrid variational principle involving  $\Psi, \rho_n, \mathbf{v}_n$ .

This paper is organized in the following manner. After a brief description of the NLKG at absolute zero, we extend it to finite temperature by introducing couplings to the normal fluid, based on Lorentz covariance. We show that the couplings can be expressed in terms of an additional nonlinear potential that has both a real and imaginary parts, and discuss its non-relativistic limit. We display quantized vorticity by transforming the scalar field theory to a global string theory. Magnus force and mutual friction are extracted in some simple examples.

## II. NLKG (NONLINEAR KLEIN-GORDON EQUATION)

In this section we use units in which  $\hbar = c = 1$ . Consider a complex scalar field  $\phi(\mathbf{x}, t)$ , which can be written in the phase representation as

$$\phi(\mathbf{x}, t) = F(\mathbf{x}, t) e^{i\sigma(\mathbf{x}, t)}. \quad (9)$$

It serves as order parameter for superfluidity through the dynamics of the phase  $\sigma(\mathbf{x}, t)$ . The physical superfluid velocity  $\mathbf{v}_s$  is related to the 4-vector

$$v^\mu = \partial^\mu \sigma \quad (10)$$

through

$$\mathbf{v}_s = \frac{\nabla \sigma}{\omega}, \quad (11)$$

Here,  $\omega$  is a frequency given by the time component of  $v^\mu$  :

$$\omega \equiv \partial^0 \sigma, \quad (12)$$

which ensures  $|\mathbf{v}_s|/c < 1$ . The Langrangian density is given by

$$\mathcal{L}_0 = g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + V, \quad (13)$$

where  $g^{\mu\nu}$  is the metric tensor. The potential  $V$  depends only on  $\phi^* \phi$ , and  $V' \equiv dV/d(\phi^* \phi)$ . The action

$$S_0 = - \int d^4x \sqrt{-g} \mathcal{L}_0 = - \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + V) \quad (14)$$

leads to the NLKG

$$(\square - V') \phi = 0, \quad (15)$$

where  $\square\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$ . Some examples of NLKG in rotating blackhole backgrounds are given in [16]. In the phase representation, the real and imaginary parts of the NLKG give rise to two hydrodynamic-like equations

$$\begin{aligned}(\square - V')F - F\nabla^\mu\sigma\nabla_\mu\sigma &= 0, \\ 2\nabla^\mu F\nabla_\mu\sigma + F\nabla^\mu\nabla_\mu\sigma &= 0.\end{aligned}\tag{16}$$

The first is the analog of the Euler equation, and the second is the continuity equation  $\nabla_\mu j_0^\mu = 0$ , where  $\nabla_\mu$  denotes the covariant derivative, with

$$j_0^\mu = \frac{1}{2i}(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) = F^2\partial^\mu\sigma\tag{17}$$

This is a number current density, corresponding to the conservation of charge  $Q = N - \bar{N}$ , where  $N, \bar{N}$  are respectively the particle and antiparticle number. Unlike the non-relativistic case, the density  $j_0^{(0)}$  is not positive definite.

If  $\sigma$  were a continuous function, we would have  $\partial^\mu v^\nu - \partial^\nu v^\mu = (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\sigma = 0$ . But  $\sigma$  is a phase angle, and only continuous modulo  $2\pi$ . The derivatives  $\partial^\mu, \partial^\nu$  do not commute when operating on  $\sigma$ , and this is the origin of quantized vorticity. Specifically, there is a class of singular solutions in which the modulus  $F$  has zeros along a space curve, the vortex line, and  $\oint_C \nabla\sigma \cdot ds = 2\pi n$  along any closed circuit  $C$  encircling the vortex line, where  $n$  is an integer.

Without going into details, we comment on the fact that the first equation in (16) can be rewritten in Euler form as an equation for  $dv/dt$ , which contains a “quantum pressure”. This pressure naturally vanishes on boundaries where  $F$  goes to zero, with a healing length. There is no need for the “Hill-Roberts pressure” [6], which is introduced by hand.

### III. NLKG WITH NORMAL FLUID: THE EFFECTIVE POTENTIAL

In the following, we consider Minkowski spacetime with  $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ . At finite temperatures, new degrees freedom arise, associated with the normal-fluid velocity field  $v_n$ . We represent the new degrees of freedom covariantly by a 4-vector  $w^\mu$ , and, following Lebedev and Khalatnikov [8], write it in a Clebsch representation of the form

$$w^\mu = \partial^\mu\alpha + \xi\partial^\mu\chi\tag{18}$$

where  $\alpha, \xi, \chi$ , are regarded as independent variables, which will be related to physical normal fluid properties. We can construct three Lorentz invariants From  $v^\mu$  and  $w^\mu$ :

$$I_1 = \frac{F^2}{2}v^\mu v_\mu, \quad I_2 = F^2v^\mu w_\mu, \quad I_3 = \frac{F^2}{2}w^\mu w_\mu\tag{19}$$

and generalize the zero-temperature Lagrangian density  $\mathcal{L}_0$  to

$$\mathcal{L} = \mathcal{L}_0 + f(I_1, I_2, I_3)\tag{20}$$

where  $f$  is a function with derivatives denoted by

$$f'_n = \frac{\partial f}{\partial I_n} \quad (21)$$

The equations of motion are obtained by varying the new action with respect to  $F, \sigma, \alpha, \xi, \chi$  :

$$\begin{aligned} \left[ \square - V' - (1 + f'_1) v_\mu v^\mu - 2f'_2 w_\mu v^\mu - f'_3 w_\mu w^\mu \right] F &= 0 \\ \partial_\mu j^\mu &= 0 \\ \partial_\mu s^\mu &= 0 \\ s^\mu \partial_\mu \xi &= 0 \\ s^\mu \partial_\mu \chi &= 0 \end{aligned} \quad (22)$$

where two current densities  $j^\mu$  and  $s^\mu$  are defined as

$$\begin{aligned} j^\mu &\equiv F^2 \left[ (2 + f'_1) v^\mu + f'_2 w^\mu \right] \\ s^\mu &\equiv F^2 (f'_2 v^\mu + f'_3 w^\mu). \end{aligned} \quad (23)$$

These are conserved current densities, identified respectively with the number current density and the entropy current density. The former reduces to  $j_0^\mu$  when  $f \equiv 0$ , and the latter is present only at nonzero temperatures. The first two equations of motion in (22) can be combined to give a new NLKG:

$$(\square - V' - W) \phi = 0 \quad (24)$$

where  $W$  is an effective potential obtained by plugging  $\phi = F e^{i\sigma}$  into (24) and then comparing with the first two equations of (22)

$$\begin{aligned} W &= f'_1 v_\mu v^\mu + 2f'_2 w_\mu v^\mu + f'_3 w_\mu w^\mu + i F^{-2} \partial_\mu (F^2 v^\mu) \\ &= f'_1 v_\mu v^\mu + 2f'_2 w_\mu v^\mu + f'_3 w_\mu w^\mu - i \left[ v^\mu \partial_\mu f'_1 + F^{-2} \partial_\mu (F^2 w^\mu f'_2) \right] / (2 + f'_1). \end{aligned} \quad (25)$$

Note that the effective potential  $W$  has both real and imaginary parts, indicating that the NLKG for the superfluid is dissipative: there is particle transfer between superfluid and normal fluid. Assuming that  $\alpha, \beta, \xi$  are continuous functions, we have  $\partial^\mu w^\nu - \partial^\nu w^\mu = \partial^\mu \xi \partial^\nu \beta - \partial^\nu \xi \partial^\mu \beta$ . Multiplying by  $s_\mu$  yields

$$s_\mu (\partial^\mu w^\nu - \partial^\nu w^\mu) = 0 \quad (26)$$

We will show that the imaginary part of the effective potential  $W$  is important in determining the mutual friction coefficients, which describe the coupling between quantized vortices and the normal fluid.

#### IV. THE NORMAL-FLUID

We represent the time and spatial components of current densities  $j^\mu, s^\mu$  as

$$\begin{aligned} j^\mu &= (\rho, \mathbf{j}) \\ s^\mu &= (s, s\mathbf{v}_n) \end{aligned} \quad (27)$$

The first relation defines total number density  $\rho$  and current  $\mathbf{j}$ , with number meaning  $N - \bar{N}$ , the difference between particle and antiparticle number. The second defines the entropy density  $s$  and the normal-fluid velocity  $\mathbf{v}_n$ . (The definition of  $\rho$  agrees with Lebedev and Khalatnikov [8], but differs from our earlier paper [1], in which the density corresponds to  $\rho \sqrt{1 - v_s^2}$  in the present convention). The conservation laws can be put in the form

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} &= 0 \\ \frac{\partial s}{\partial t} + \nabla \cdot (s \mathbf{v}_n) &= 0\end{aligned}\tag{28}$$

The second equation of (23) can be rewritten as

$$w^\mu = \frac{s^\mu}{F^2 f'_3} - \frac{f'_2}{f'_3} v^\mu\tag{29}$$

Substitution into the first equation of (23) gives

$$j^\mu = F^2 \left( 2 + f'_1 - \frac{f'^2_2}{f'_3} \right) v^\mu + \frac{f'_2}{f'_3} s^\mu\tag{30}$$

whose components are given by

$$\begin{aligned}\rho &= F^2 \left( 2 + f'_1 - \frac{f'^2_2}{f'_3} \right) \omega + \frac{f'_2}{f'_3} s \\ \mathbf{j} &= F^2 \left( 2 + f'_1 - \frac{f'^2_2}{f'_3} \right) \omega \mathbf{v}_s + \frac{f'_2}{f'_3} s \mathbf{v}_n\end{aligned}\tag{31}$$

We make the identification

$$\begin{aligned}\rho_s &= \omega F^2 \left( 2 + f'_1 - \frac{f'^2_2}{f'_3} \right) \\ \rho_n &= \frac{f'_2}{f'_3} s\end{aligned}\tag{32}$$

such as to give

$$\begin{aligned}\rho &= \rho_s + \rho_n \\ \mathbf{j} &= \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n\end{aligned}\tag{33}$$

We can write, in manifestly covariant form,

$$j^\mu = \frac{\rho_s}{\omega} v^\mu + \frac{\rho_n}{s} s^\mu\tag{34}$$

The energy-momentum tensor of the superfluid is given by

$$\begin{aligned}T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu F)} \partial^\nu F + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \sigma)} \partial^\nu \sigma + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \alpha)} \partial^\nu \alpha + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \beta)} \partial^\nu \beta - g^{\mu\nu} \mathcal{L} \\ &= 2 \partial^\mu F \partial^\nu F + j^\mu v^\nu + s^\mu w^\nu - g^{\mu\nu} \mathcal{L}\end{aligned}\tag{35}$$

Using (23), we obtain the symmetric form

$$T^{\mu\nu} = 2\partial^\mu F \partial^\nu F + \frac{\rho_s}{\omega} v^\mu v^\nu + \frac{\rho_n}{\tilde{s}} s^\mu s^\nu - g^{\mu\nu} \mathcal{L} \quad (36)$$

where  $\tilde{s} \equiv \rho_n F^2 f'_3$ . The spatial components give the stress-energy tensor

$$T^{jk} = 2\partial^j F \partial^k F + \omega \rho_s v_s^j v_s^k + (s^2/\tilde{s}) \rho_n v_n^j v_n^k - \delta^{jk} \mathcal{L} \quad (37)$$

where we have used (11) and (27). The coefficient of  $v_n^j v_n^k$  should be  $\omega \rho_n$ , as suggested by comparison with (4). Thus we have the relation

$$s^2 = \omega \tilde{s} \quad (38)$$

Using (38) and (32), we can now express the parameters  $f'_n$  in terms of observable normal-fluid properties:

$$\begin{aligned} f'_1 &= \frac{\rho}{\rho_0} - 2 \\ f'_2 &= \frac{s}{\rho_0} \\ f'_3 &= \frac{s^2}{\rho_0 \rho_n} \end{aligned} \quad (39)$$

where  $\rho_0$  is the density at absolute zero temperature:

$$\rho_0 \equiv F^2 \omega \quad (40)$$

The NLKG describes the superfluid and its coupling to the normal fluid, whose dynamics requires separate treatment. A general hydrodynamic treatment based on conservation laws and thermodynamics has been given by Lebedev and Khalatnikov [8] and by Carter and Khalatnikov [12]. A general action principle has been discussed by Geurst [7]. A more detailed description of the normal fluid will depend on the specific model. In this respect, a relativistic ideal gas model of the normal fluid may be found in [13], and a treatment based on quantum field theory of the scalar field is given in [17].

Note that the physical meaning of the currents and densities depend on the reference frames and we compare our results with [13]. We consider the following Lorentz scalars:

$$\begin{aligned} v^\mu v_\mu &= -c^2 \mu^2 \\ s^\mu s_\mu &= -c^2 s^2 \\ v^\mu s_\mu &= -c^2 y^2 \end{aligned} \quad (41)$$

Our choices for  $v^\mu$  and  $S^\mu$  are

$$\begin{aligned} v^\mu &= (\tilde{\mu}, \nabla \sigma) = \tilde{\mu}(1, \mathbf{v}_s) \\ s^\mu &= \tilde{s}(1, \mathbf{v}_n). \end{aligned} \quad (42)$$

(Here we changed some notations for comparison purpose. We defined before in the equations (10), (11) and (12)

$$v^\mu = \partial^\mu \sigma, \quad \mathbf{v}_s = \frac{\mathbf{v}}{\omega} = \frac{\nabla \sigma}{\omega}, \quad \omega \equiv \partial^0 \sigma, \quad (43)$$



and we have changed  $\omega$  to  $\tilde{\mu}$ , and  $s$  to  $\tilde{s}$  respectively.) Note that the velocities  $\mathbf{v}_s$  and  $\mathbf{v}_n$  are defined in the lab frame. From (41) we obtain

$$\begin{aligned}\tilde{\mu} &= \gamma_s \mu, \quad \gamma_s \equiv \frac{1}{\sqrt{1 - \frac{\mathbf{v}_s^2}{c^2}}}, \\ \tilde{s} &= \gamma_n s, \quad \gamma_n \equiv \frac{1}{\sqrt{1 - \frac{\mathbf{v}_n^2}{c^2}}}.\end{aligned}\tag{44}$$

While  $\mu, s$  are Lorentz scalars,  $\tilde{\mu}, \tilde{s}$  are not due to the  $\gamma$ -factors in the above equations. The crossing term yields

$$v^\mu s_\mu \equiv -c^2 y^2 = -c^2 \tilde{\mu} \tilde{s} \left(1 - \frac{\mathbf{v}_s \cdot \mathbf{v}_n}{c^2}\right) = -c^2 \mu s \gamma_n \gamma_s \left(1 - \frac{\mathbf{v}_s \cdot \mathbf{v}_n}{c^2}\right).\tag{45}$$

We define a relative velocity,  $\mathbf{v}_{ns}$ , between the superfluid velocity  $\mathbf{v}_s$  and the normal velocity  $\mathbf{v}_n$  according to the relativistic velocity-addition formula

$$\mathbf{v}_{ns} = \frac{\mathbf{v}_n - \mathbf{v}_s}{1 - \frac{\mathbf{v}_s \cdot \mathbf{v}_n}{c^2}},\tag{46}$$

and then find

$$1 - \frac{\mathbf{v}_{ns}^2}{c^2} = \frac{1}{\gamma_n^2 \gamma_s^2 \left(1 - \frac{\mathbf{v}_s \cdot \mathbf{v}_n}{c^2}\right)^2}.\tag{47}$$

Plugging into (45) we obtain

$$\frac{\mathbf{v}_{ns}^2}{c^2} = 1 - \frac{\mu^2 s^2}{y^4}\tag{48}$$

which is exactly the relative translation speed between the “normal” and “superfluid” frames used in Ref. [13]. One can identify the relativistic generalization of the mass densities of the superfluid and normal fluids,  $\hat{\rho}_n$  and  $\hat{\rho}_s$ , by considering the decomposition of the stress-energy tensor  $T^{\mu\nu}$ . The advantage of finding  $\hat{\rho}_n$  and  $\hat{\rho}_s$  based on the decomposition of  $T^{\mu\nu}$  is that it avoids the use of “rest mass”, more precisely, the question on which frame should be considered as the rest frame of the fluid element. What is needed is the relative translation velocity between the normal fluid and the superfluid frames [13]. Note that

$$y^2 = \mu_n s = \mu s_s, \quad \text{where } \mu_n \equiv \mu / \sqrt{1 - \frac{\mathbf{v}_{ns}^2}{c^2}}, \quad s_s \equiv s / \sqrt{1 - \frac{\mathbf{v}_{ns}^2}{c^2}}\tag{49}$$

where  $s_s$  is the entropy density defined in the superfluid frame and the  $\mu_n$  is the chemical potential defined in the normal fluid frame. The stress-energy tensor becomes

$$\begin{aligned}T^{\mu\nu} &= 2\partial^\mu F \partial^\nu F + \frac{\hat{\rho}_s}{\mu_n^2} v^\mu v^\nu + \frac{\hat{\rho}_n}{s_s^2} s^\mu s^\nu - g^{\mu\nu} \mathcal{L} \\ \hat{\rho}_s &\equiv \mu_n^2 F^2 \left[ 2 + f'_1 - \frac{(f'_2)^2}{f'_3} \right] \\ \hat{\rho}_n &\equiv s_s^2 \frac{1}{F^2 f'_3}.\end{aligned}\tag{50}$$

It is not hard to check that these are consistent with the results in Ref.[13]. In fact the explicit correspondences are

$$\begin{aligned}\mathcal{A} &= -\frac{\omega}{s} \frac{\rho_n}{\rho_s} \\ \mathcal{B} &= \frac{\omega}{\rho_s} \\ C &= \frac{\omega \rho_n}{s^2} \left( 1 + \frac{\rho_n}{\rho_s} \right)\end{aligned}\tag{51}$$

where  $\mathcal{A}, \mathcal{B}, C$  are quantities defined in [13]

$$\begin{aligned}w^\mu &= C s^\mu + \mathcal{A} j^\mu, \\ v^\mu &= \mathcal{A} s^\mu + \mathcal{B} j^\mu,\end{aligned}\tag{52}$$

## V. THE NON-RELATIVISTIC LIMIT: MODIFIED NLSE

A solution  $\phi$  of the NLKG contains both positive and negative frequencies. It approaches the nonrelativistic limit when one sign (say, positive) becomes dominant. Formally, we write

$$\phi \xrightarrow{c \rightarrow \infty} \Psi e^{-i(mc^2/\hbar)t}\tag{53}$$

where  $m$  is a large mass scale. The nonrelativistic wave function  $\Psi$  can be represented in the form

$$\Psi = \sqrt{\rho} e^{i\beta}\tag{54}$$

where  $\rho$  is the non-relativistic superfluid density, and

$$\mathbf{v}_s = \frac{\hbar}{m} \nabla \beta\tag{55}$$

is the non-relativistic superfluid velocity. The nonrelativistic phase  $\beta$  is related to the phase  $\sigma$  of the relativistic scalar field  $\phi$  through

$$\dot{\beta} = \dot{\sigma} + \frac{mc^2}{\hbar}, \quad \nabla \beta = \nabla \sigma.\tag{56}$$

The wave function  $\Psi$  satisfies an NLSE (nonlinear Schrödinger equation) (see [1] for details). To derive it, it is easier to start from the Lagrangian. Let  $\mathcal{L}_0$  be the nonrelativistic Lagrangian density at absolute zero, which leads to an NLSE with cubic nonlinearity. We show how the normal fluid may be introduced, following Coste [15], but reformulated from our point of view.

Let the superfluid density and current density be denoted respectively by  $\rho$  and  $\mathbf{j} = \rho \mathbf{v}_s$ . The degrees of freedom  $\rho_n, \mathbf{v}_n$  associated with the normal fluid can be introduced via gauge-like potentials  $\varphi, \mathbf{A}$ , through the transformation  $\frac{\partial}{\partial t} \rightarrow \nabla - i\varphi, \nabla \rightarrow \nabla - i\mathbf{A}$ . This method was apparently first suggested by Bogoliubov in an unpublished note [14]. The Lagrangian density at finite temperatures is

$$\mathcal{L} = \mathcal{L}_0 + \rho \varphi - \mathbf{j} \cdot \mathbf{A} + \frac{1}{2} \rho A^2\tag{57}$$

This is not locally gauge-invariant, (and had better not be, for otherwise the gauge transformation would have no physical effect.) The term  $\frac{1}{2}\rho A^2$ , while crucial for local gauge invariance, is irrelevant here, and will be dropped. Using arguments based on Galillean covariance, Coste [15] writes

$$\begin{aligned} \mathbf{A} &= \alpha (\mathbf{v}_s - \mathbf{v}_n) \\ \varphi &= \mathbf{v}_n \cdot \mathbf{A} \end{aligned} \quad (58)$$

where  $\alpha$  is a scalar function, and  $\mathbf{v}_n$  is the normal-fluid velocity. The equations of motion are then obtained through the action principle. We omit details and just cite the final result. Assuming an original NLSE with quartic nonlinearity, we obtain a modified equation

$$\begin{aligned} i \frac{\partial \Psi}{\partial t} &= -\frac{1}{2} \nabla^2 \Psi + (|\Psi|^2 - 1 + U) \Psi \\ U &= -\frac{1}{2} (\mathbf{v}_n - \mathbf{v}_s)^2 \frac{\partial \rho_n}{\partial \rho} - \frac{i}{2\rho} \nabla \cdot [\rho_n (\mathbf{v}_n - \mathbf{v}_s)] \end{aligned} \quad (59)$$

where  $\hbar = m = 1$ , and all coupling parameters have been scaled to unity. The normal fluid enters via the effective potential  $U$ , which vanishes at absolute zero. The real part of  $U$  contributes to the phase change of  $\Psi$ , and thus to superfluid flow. The imaginary part contributes to  $\dot{\Psi}$ , rendering  $\int d^3x |\Psi|^2$  non-conserved, signifying particle transfer between superfluid and normal fluid.

## VI. FROM PHENOMENOLOGY OF QUANTIZED VORTICITY TO NLKG FORMULATION

A phenomenological treatment of quantized vorticity in the non-relativistic domain was pioneered by Schwarz [21] in the non-relativistic domain, based on the following physical picture (see Appendix A for notations). A vortex configuration is characterized by a space curve called the vortex line, described by the position vector  $\mathbf{s}(\xi, t)$ , where  $\xi$  is a parameter that runs along the line. The vortex line may be made up of disjoint closed loops, and curves that terminate on boundaries. The parameter  $\xi$  run through all of the components according to some convention. The superfluid density vanishes on the vortex line with a characteristic healing length. We can picture the vortex line as a tube with effective radius  $a_0$  of the order of the healing length. This core size is supposed to be much smaller than any other length in the theory. When we refer a point on the vortex line, we mean some point within the core. Let  $\mathbf{s}' \equiv \partial \mathbf{s} / \partial \xi$ . The triad  $\mathbf{s}, \mathbf{s}', \mathbf{s}''$  gives a local orthogonal coordinate system. The local radius of curvature is given by  $R = |\mathbf{s}''|^{-1}$ .

The superfluid velocity is determined up to a potential flow by the equation

$$\nabla \times \mathbf{v}_s = \boldsymbol{\kappa} \quad (60)$$

where  $\boldsymbol{\kappa}(\mathbf{r}, t)$  is the vorticity density, which is nonvanishing only on the vortex line:

$$\boldsymbol{\kappa}(\mathbf{r}, t) = \kappa_0 \int d\xi \delta(\mathbf{r} - \mathbf{s}(\xi, t)) \quad (61)$$

We decompose the superfluid velocity into an irrotational part  $\mathbf{v}_0$ , and a rotational part  $\mathbf{b}$ :

$$\begin{aligned} \mathbf{v}_s &= \mathbf{v}_0 + \mathbf{b} \\ \nabla \times \mathbf{v}_0 &= \nabla \cdot \mathbf{b} = 0 \\ \nabla \times \mathbf{b} &= \boldsymbol{\kappa} \end{aligned} \quad (62)$$

The velocity field  $\mathbf{b}$  is like a magnetic field produced by the current density  $\kappa$ , and is given by the Biot-Savart law

$$\mathbf{b}(\mathbf{r}, t) = \frac{\kappa_0}{4\pi} \int \frac{(\mathbf{s}_1 - \mathbf{r}) \times d\mathbf{s}_1}{|\mathbf{s}_1 - \mathbf{r}|^3} \quad (63)$$

where  $\mathbf{s}_1$  is a particular point on the vortex line.

The velocity of the vortex line at any point is influence by the shape of the entire vortex line. In a “local inducting approximation” (LIA), one considers only the effects from the immediate neighborhood of the point. In this case, this the local velocity is the translational velocity of an osculating vortex ring at that point, which is normal to the plane of the vortex ring, and approximately inversely proportional to its radius of curvature  $R$ . For a vortex line at absolute zero, this leads to the equation

$$\begin{aligned} \dot{\mathbf{s}}_0 &= \beta \mathbf{s}' \times \mathbf{s}'' + \mathbf{v}_s \\ \beta &= \frac{\kappa_0}{4\pi} \ln \left( \frac{c_0 \bar{R}}{a_0} \right) \end{aligned} \quad (64)$$

where  $\dot{\mathbf{s}}_0 \equiv \partial \mathbf{s}_0 / \partial t$ ,  $\bar{R}$  is the average radius of curvature, and  $c_0$  is a constant of order unity.

At finite temperatures, there is a normal fluid, which exerts a dissipative force per unit length  $\mathbf{f}_D$  on the vortex line. It can be fit phenomenologically by the formula

$$\frac{\mathbf{f}_D}{\rho_s \kappa_0} = -\alpha \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_{ns} - \mathbf{v}_{sl})] - \alpha' \mathbf{s}' \times (\mathbf{v}_{ns} - \mathbf{v}_{sl}) \quad (65)$$

where  $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ , and  $\alpha, \alpha'$  are temperature-dependent parameters. The vortex line experiences a Magnus force per unit length  $\mathbf{f}_M$ , when the vortex line velocity  $\mathbf{v}_L(\xi, t) \equiv \dot{\mathbf{s}}(\xi, t)$  is different from the local superfluid velocity  $\mathbf{v}_{sl}(\xi, t) \equiv \mathbf{v}_s(\mathbf{s}(\xi, t), t)$ :

$$\frac{\mathbf{f}_M}{\rho_s \kappa_0} = \mathbf{s}' \times (\mathbf{v}_L - \mathbf{v}_{sl}) \quad (66)$$

The phenomenological equations give physical insight, but for actual computations it is simpler to use the NLKG directly. As shown in [1], complex phenomena such as vortex formation and reconnection can be exhibited in numerical solutions of the NLKG. When quantum vorticity appears, the phase  $\sigma$  of the complex field cannot be smooth everywhere, hence  $\nabla_\mu v_\nu - \nabla_\nu v_\mu \neq 0$ . From numerical calculations, the phase  $\sigma$  has ambiguity and the modulus  $F$  vanishes at the locations of vortices. To be consistent with the NLKG with effective potential describing the normal fluid effects, the variational principle should be applied to the Lagrangian

$$\mathcal{L}(f_\mu, v_\mu, w_\mu) = \mathcal{L}(F, \nabla_\mu F, \nabla_\mu \sigma, \nabla_\mu \alpha, \zeta, \nabla_\mu \beta) \quad (67)$$

where  $f_\mu \equiv \nabla_\mu F$  and  $\nabla \sigma$  is generally NOT a smooth function. Note that one can split the Lagrangian into

$$\mathcal{L} = \mathcal{L}_{\text{NLKG}}^0 + \mathcal{L}^T \quad (68)$$

where  $\mathcal{L}_{\text{NLKG}}^0$  is similar to the zero-temperature cases

$$\mathcal{L}_{\text{NLKG}}^0 = -g^{\mu\nu} \partial_\mu F \partial_\nu F - F^2 g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(F^2), \quad (69)$$

while  $\mathcal{L}^T$  includes finite-temperature effects. Following similar notations in [8], we rewrite

$$\begin{aligned} v_\mu &= \nabla_\mu \sigma \equiv \nabla_\mu \varphi + b_\mu, \\ w_\mu &= \nabla_\mu \alpha + \zeta \nabla_\mu \beta, \end{aligned} \quad (70)$$

where  $\varphi$  is a smooth function whose gradient is curl-free, i.e.,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \varphi = 0, \quad (71)$$

while the vector field  $b_\mu$  gives the vorticity, described by its “field strength”  $b_{\mu\nu}$

$$\nabla_\mu v_\nu - \nabla_\nu v_\mu = \nabla_\mu b_\nu - \nabla_\nu b_\mu \equiv b_{\mu\nu}. \quad (72)$$

Now both the stress-energy tensor and the mass current should include the contribution of the vorticity. For simplicity, let us first consider the zero temperature  $T = 0$  cases without  $w^\mu$  as in [8]

$$T^\mu_\nu = -\frac{\partial \mathcal{L}}{\partial v_\mu} v_\nu - 2 \frac{\partial \mathcal{L}}{\partial b_{\mu\tau}} b_{\nu\tau} + \delta^\mu_\nu \mathcal{L} \quad (73)$$

$$j^\mu = -\frac{\partial \mathcal{L}}{\partial v_\mu} - 2 \nabla_\tau \frac{\partial \mathcal{L}}{\partial b_{\mu\tau}}, \quad (74)$$

and the conservation law  $\nabla_\mu T^\mu_\nu = 0$  leads to

$$j^\mu b_{\mu\nu} = 0, \quad (75)$$

in comparison with the curl-free or irrotational cases in which  $b_{\mu\nu} = 0$ . Note that  $j^\mu$  also has a vorticity dependence. Therefore after the decomposition (70), besides  $b_\mu$  the Lagrangian should also contain  $b_{\mu\nu}$  explicitly. From the zero temperature example given in [8], the inclusion of  $b_\mu$  and  $b_{\mu\nu}$  in the Lagrangian is much more complicated than having a “kinetic” term  $\sim b^{\mu\nu} b_{\mu\nu}$  as one may have imagined, in analogy to the electromagnetic case. This will be discussed in the next section.

## VII. STRING THEORY OF QUANTIZED VORTICITY

We give a relativistically covariant description of quantized vorticity in Minkowski spacetime with metric  $\text{diag}(-1, 1, 1, 1)$ . It is easily generalized to curved spacetime. The vortex configuration is specified by a space curve, which sweeps out a world sheet in 4D spacetime. The dynamics of the vortex line is therefore that of a relativistic string, which has been widely discussed in the literature [18] [19]. We summarize known results from our perspective.

To begin, we covariantly separate rotational flow from irrotational flow by writing

$$v^\mu \equiv \partial^\mu \sigma = \partial^\mu \chi + b^\mu \quad (76)$$

where  $\chi$  is a continuous function (whereas  $\sigma$  is only continuous modulo  $2\pi$ ), and  $b^\mu$  describes vorticity. We define a “smooth” order parameter  $\psi$ , with the phase  $\chi$  :

$$\psi = F e^{i\chi} \quad (77)$$

The Lagrangian density can be rewritten in terms of  $\psi$  and  $b^\mu$  :

$$\mathcal{L}_0 = \partial^\mu \phi^* \partial_\mu \phi + V(\phi^* \phi) = (\partial^\mu + i b^\mu) \psi^* (\partial_\mu - i b_\mu) \psi + V(\psi^* \psi) \quad (78)$$

This says that we can start with potential flow described by  $\psi$ , and introduce vorticity through by introducing a “gauge field”  $b^\mu$ . The system is invariant under a local “gauge transformation”  $\psi \rightarrow \psi'$ ,  $b_\mu \rightarrow b'_\mu$ , with

$$\begin{aligned} \psi' &= e^{-i\alpha} \psi \\ b'_\mu &= b_\mu + \partial_\mu \alpha \end{aligned} \quad (79)$$

where  $\alpha(x)$  is a continuous function; the transformation suggests an emergent “gauge symmetry” and is equivalent to a shift  $\chi \rightarrow \chi - \alpha$ . Note that  $\mathbf{b}$  has a dual personality: on the one hand, it is like a magnetic field according to (63), and on the other hand it is like a gauge field in the present context. Note that  $b_\mu$  is constrained by the vortex quantization condition, which can be represented covariantly as

$$\oint_C dx^\mu b_\mu = 2\pi n \quad (n = 0, \pm 1, 2, \dots) \quad (80)$$

By means of the Stokes theorem, we can rewrite this as

$$\frac{1}{2} \int_S dS^{\mu\nu} b_{\mu\nu} = 2\pi n \quad (n = 0, \pm 1, 2, \dots) \quad (81)$$

where  $S$  is a surface bounded by the closed path  $C$ ,  $dS^{\mu\nu}$  is a surface element, and  $b^{\mu\nu}$  is the antisymmetric vorticity tensor defined by

$$b^{\mu\nu} \equiv \partial^\mu v^\nu - \partial^\nu v^\mu = \partial^\mu b^\nu - \partial^\nu b^\mu = [\partial^\mu, \partial^\nu] \sigma \quad (82)$$

Now we define the “dual” of vorticity  $b_{\mu\nu}$  [18]

$$\tilde{b}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\tau} b_{\rho\tau} \quad (83)$$

In terms of the phase  $\sigma$ , it becomes

$$\tilde{b}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\tau} [\partial_\rho, \partial_\tau] \sigma. \quad (84)$$

The dual vorticity  $\tilde{b}^{\mu\nu}$  is a distribution, e.g. for a static, straight vortex line lying on the  $z$ -axis,

$$\tilde{b}^{03} = \frac{1}{2} \delta(x) \delta(y) \quad (85)$$

In general,  $\tilde{b}^{\mu\nu}$  perform a projection onto the worldsheet swept by the vortex line. Suppose the worldsheet is parametrized by  $x^\mu = x^\mu(\zeta^a)$ , ( $a = 0, 1$ ),  $\tilde{b}^{\mu\nu}$  can be written as [18]

$$\tilde{b}^{\mu\nu} = \frac{1}{2} \int \delta^{(4)}(x - x(\zeta^a)) d\sigma^{\mu\nu} \quad (86)$$

where  $d\sigma^{\mu\nu} \equiv \epsilon^{ab} x_{,a}^\mu x_{,b}^\nu d^2\zeta$  is the area element of the worldsheet.

These are necessary ingredients connecting the (modified) NLKG to its intrinsic vortex dynamics (or connecting a field theory to a “string” theory). From equations (84) and (86) it is clear that the vortex dynamics is determined by the phase  $\sigma$  and the zero of the modulus  $F$ .

The vector  $b_\mu$  can be related to a Kalb-Ramond potential similar to [18]. The decomposition

$$v_\mu = \partial_\mu \varphi + b_\mu \equiv u_\mu + b_\mu \quad (87)$$

with the identification

$$b_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} \partial^\nu B^{\lambda\rho} \quad (88)$$

where  $B^{\mu\nu}$  is the antisymmetric Kalb-Ramond field [20], generalizes the usual Helmholtz decomposition for a 3-vector  $\mathbf{v}$ ,

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp, \quad \nabla \times \mathbf{v}_\parallel = \nabla \cdot \mathbf{v}_\perp = 0. \quad (89)$$

It is easy to see that in equation (87) the “longitudinal” component  $u_\mu = \partial_\mu \varphi$  does not contribute to the vorticity while the “transverse” component  $b_\mu = 1/2 \epsilon_{\mu\nu\lambda\rho} \partial^\nu B^{\lambda\rho}$  is divergenceless for a regular Kalb-Ramond field  $B^{\mu\nu}$ . To see why the relation (88) is the relativistic generalization of the three-dimensional analogue

$$\mathbf{v}_\perp = \nabla \times \mathbf{A}, \quad (90)$$

one may set the components of the Kalb-Ramond field to be  $(i, j, k = 1, 2, 3)$

$$B_{i0} = A_i, \quad B_{ij} = \epsilon_{ijk} x^k \quad (91)$$

and then obtain

$$b^i = \epsilon_{ijk} \partial_j A_k, \quad b^0 = \text{const.} \quad (92)$$

The relativistic generalization of the three-dimensional vorticity vector

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}, \quad (93)$$

is in a Chern-Simons form [8],

$$K^\mu = \epsilon^{\mu\nu\rho\tau} b_\nu \partial_\rho b_\tau \quad (94)$$

whose spatial component contains a term

$$\epsilon^{i0jk} b_0 \partial_j b_k = -b_0 \epsilon^{ijk} \partial_j b_k. \quad (95)$$

Therefore, the role of the vector potential in  $\mathbf{v} = \nabla \times \mathbf{A}$  is played by the Kalb-Ramond field  $B^{\mu\nu}$  and the role of the vorticity field  $\boldsymbol{\omega} = \nabla \times \mathbf{v}$  is played by the Chern-Simons form  $K^\mu = \epsilon^{\mu\nu\rho\tau} b_\nu \partial_\rho b_\tau$ .

Once we have identified  $b_\mu = 1/2 \epsilon_{\mu\nu\lambda\rho} \partial^\nu B^{\lambda\rho}$ , we can use the Kalb-Ramond action [20] as the effective action for a vortex line or a vortex ring. With the Kalb-Ramond field strength

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu} \quad (96)$$

the original nonlinear action for the Klein-Gordon field  $\Phi = F e^{i\sigma}$  (at  $T = 0$ ) becomes

$$S_0[F, \sigma] \rightarrow S_0[F, B_{\mu\nu}] = \int d^4x \left[ \partial^\mu F \partial_\mu F + \frac{1}{6F^2} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - V(F^2) \right] + 2\pi \int B_{\mu\nu} d\sigma^{\mu\nu}. \quad (97)$$

Integrating over the massive  $F$  modes for a string solution gives the Kalb-Ramond action [18].

Now we look back at the original Lagrangian and consider its independent variables: We have  $v^2 = v^\mu v_\mu$ ,  $\omega^2 = \omega_{\mu\nu} \omega^{\mu\nu}$  with  $\omega_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu$ , and  $h^2 = h_\mu h^\mu$ , where the Chern-Simon current  $h^\mu$  is defined as (similar to Eq. (94))

$$h^\mu = \epsilon^{\mu\nu\rho\sigma} \omega_{\nu\rho} v_\sigma. \quad (98)$$

It is easy to see that

$$h^2 = -\frac{1}{2}(v^2 \omega^2 + 2v_\mu v^\nu \omega_{\nu\lambda} \omega^{\lambda\mu}) \quad (99)$$

therefore we see that  $v^2, \omega^2, h^2$  can be considered as independent variables in the Lagrangian. One can easily write down other Lorentz invariants such as

$$h^\mu v^\nu \omega_{\mu\nu}, \quad h^\mu v^\nu \tilde{\omega}_{\mu\nu}, \quad \tilde{\omega}_{\mu\nu} \tilde{\omega}^{\mu\nu}, \quad \omega_{\mu\nu} \tilde{\omega}^{\mu\nu} \quad (100)$$

where the dual tensor  $\tilde{\omega}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\tau} \omega_{\rho\tau}$ . However, with the help of two identities

$$\begin{aligned} \omega_{\mu\lambda} \tilde{\omega}^{\lambda\nu} &= -\frac{1}{4} \delta_\mu^\nu \omega_{\rho\tau} \tilde{\omega}^{\rho\tau}, \\ \tilde{\omega}_{\mu\lambda} \tilde{\omega}^{\lambda\nu} &= \omega_{\mu\lambda} \omega^{\lambda\nu} + \frac{1}{2} \delta_\mu^\nu \omega_{\rho\tau} \omega^{\rho\tau}, \end{aligned} \quad (101)$$

one can show that what is really important is the  $\omega_{\mu\nu} \tilde{\omega}^{\mu\nu}$  term and the other terms can be reduced to combinations of known terms. Also, note that

$$\omega_{\mu\nu} \tilde{\omega}^{\mu\nu} = \frac{1}{2} \partial_\mu h^\mu \quad (102)$$

similar to the gauge theory cases where  $F\tilde{F} \sim \partial_\mu K^\mu$ , i.e. the topological charge term can be written as the divergence of the Chern-Simons current. Therefore the Lagrangian should only depend on these Lorentz scalars

$$\mathcal{L}_{T=0} = \mathcal{L}(v^2, \omega^2, h^2, \omega\tilde{\omega}). \quad (103)$$

For the finite temperature cases,  $w_\mu$  should be included as well. Neglecting classical vorticity, one can write

$$\mathcal{L}^T = \mathcal{L}(v^2, \omega^2, h^2, w^2, v \cdot w, \omega\tilde{\omega}). \quad (104)$$

Note that in the present paper we are not intended to write down an explicit Lagrangian for  $\mathcal{L}^T$ . Instead we aimed to show how the relevant degrees of freedom come from the (modified) NLKG/NLSE. In practice what we propose to solve numerically is the original (modified) NLKG/NLSE, similar to what we have done in [1].



### VIII. MAGNUS FORCE, MUTUAL FRICTION AND OTHER FORCES

Just like the vorticity is built in the NLKG, forces like the Magnus force, the mutual friction between the quantum vorticity and the normal fluid should be included automatically in the effective potential part of the NLKG. Solving such a NLKG should yield all the effects these forces produce. First let us consider the Magnus force. Any force in vortex dynamics is connected with some velocity by the Magnus relation, connecting the vortex line velocity and the external force per unit length applied to the vortex line (see e.g. [9])

$$\mathbf{F} = \rho \boldsymbol{\kappa} \times (\mathbf{v}_0 - \mathbf{v}_L) \quad (105)$$

where  $\boldsymbol{\kappa}$  is the circulation vector of magnitude  $\kappa$ ,  $\mathbf{v}_0$  is the constant velocity that a fluid current passes the vortex line and  $\mathbf{v}_L$  is the velocity of the vortex line. We use a simple example at zero-temperature to demonstrate the existence of the Magnus force. From the decomposition  $v_\mu = \partial_\mu \varphi + b_\mu$ , we take

$$\varphi = \omega t, \quad (\omega \text{ is a constant}) \quad (106)$$

which corresponds to a constant background

$$H_{ijk}^0 \propto \epsilon_{ijk} \quad (107)$$

in terms of the Kalb-Ramond field. With proper gauge the equation of motion of string is [18]

$$\mu_0(\ddot{q}_\mu - q''_\mu) = 4\pi F_0(H_{\mu\nu\lambda}^0 + \dots)\dot{q}^\nu q'^\lambda \quad (108)$$

Its spatial component gives

$$\mu_0(\ddot{q}_i - q''_i) = 4\pi F_0 \epsilon_{ijk} \dot{q}^j q'^k \quad (109)$$

Note that  $\dot{q}^j$  is the vortex line velocity  $\mathbf{v}_L$  and  $\mathbf{q}'$  is the vortex line tangent. We see that the right-hand side of the above equation can be written as  $\mathbf{v}_L \times \mathbf{q}'$ , which shows the existence of Magnus force in the superfluid rest frame

$$\mathbf{F}_M \sim \mathbf{v}_L \times \boldsymbol{\kappa} \quad (110)$$

Now we consider the mutual friction between quantum vortices and the normal flow. Notice that there is a non-vanishing imaginary part in the effective potential  $W$  in Eq. (25). For simplicity we assume that this imaginary part is a constant  $\propto \gamma$ . At the non-relativistic limit it reduces to a damped NLSE

$$(i - \gamma)\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + \lambda |\psi|^2 - \mu \right) \psi \quad (111)$$

which for small  $\gamma$  becomes approximately [23]

$$i\hbar \frac{\partial \psi}{\partial t} = (1 - i\gamma) \frac{\delta H[\psi, \psi^*]}{\delta \psi^*} \quad (112)$$

where  $H[\psi, \psi^*]$  is the Gross-Pitaevskii energy functional

$$H[\psi, \psi^*] = \int d^3x \left[ \frac{\hbar^2}{2m} |\nabla \psi|^2 - \mu |\psi|^2 + \frac{\lambda}{2} |\psi|^4 \right] \quad (113)$$

To connect the order parameter  $\psi(\mathbf{x}, t)$  to the motion of the vortex line, we follow [22, 24] and consider  $\psi(\mathbf{x}, t)$  as a functional of the vortex configuration  $\psi(\mathbf{x}, \mathbf{s}(\xi, t))$ . The time-derivative of  $\psi(\mathbf{x}, t)$ , for example, is then related to the vortex line velocity  $\dot{\mathbf{s}}$  as

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = \int_C \frac{\delta \psi(\mathbf{x}, t)}{\delta \mathbf{s}(\xi', t)} \frac{\partial \mathbf{s}(\xi', t)}{\partial t} d\xi' \quad (114)$$

and then it has been shown in [24] that

$$\dot{\mathbf{s}} = \frac{1 + \gamma^2}{1 + \beta^2 \gamma^2} \mathbf{b} + \frac{\beta \gamma (1 + \gamma^2)}{1 + \beta^2 \gamma^2} \mathbf{s}' \times \mathbf{b} \quad (115)$$

where the constant  $\beta$  is defined in Eq. (64). This allows identifying the mutual friction coefficients [24]

$$\alpha = \frac{\beta \gamma (1 + \gamma^2)}{1 + \beta^2 \gamma^2}, \quad \alpha' = \frac{(\beta^2 - 1) \gamma^2}{1 + \beta^2 \gamma^2} \quad (116)$$

## IX. CONCLUSIONS AND DISCUSSIONS

In this paper we studied relativistic two-fluid model with quantized vorticity via a modified NLKG. An effective potential is introduced to describe the coupling of the superfluid and the normal fluid. It has been shown that such a formulation incorporates vorticity and the related vortex dynamics, and hence, can facilitate numerical analysis which is usually quite complicated from phenomenological point of view (e.g. Schwarz's numerical studies based on vortex filaments). We also considered the connections to other formulations, especially the duality between scalar field and Kalb-Ramond field, and the similarity between quantized vortices and global strings. We propose that just like in the zero-temperature pure superfluid cases (as we have shown numerically in [1]), quantum vorticity and quantum turbulence should be studied using the modified NLKG/NLSE, possibly coupled to other equations depending on the systems or circumstances, in a relativistic or non-relativistic way.

What is the range of validity of the modified NLKG/NLSE? This is a interesting question, with different answers from different points of view. One can derive the NLKG/NLSE from the quantum N-body wave function, but the validity of this approach is limited to weak interparticle interactions, and in the neighborhood of the ground state of the system, i.e., at low temperatures. The reason is as follows. First, the assumption that the interparticle potential is a delta function can be justified only for weak interactions described through a small S-wave scattering length, which give the equivalent hard-sphere interaction. Secondly, the derivation from the quantum N-body problem corresponds to a mean-field approximation, in which one assumes that effects from excitations from the ground state are small. From this point of view, then, the NLKG/NLSE is a weak-interaction low-temperature approximation.

In the Ginsburg-Landau theory of phase transitions governed by an order parameter, on the other hand, the NLKG/NLSE is a purely phenomenological equation, valid near the transition point of the phase transition. Thus, the order parameter is assumed to be small. One expands the nonlinear potential in powers of the order parameters, and just retain the first few terms. From this point of view, the NLKG/NLSE is valid in the neighborhood of the phase transition that creates the order parameter. Which view one adopts would depend on the application.

### Appendix A: Notations on phenomenology of vortex filaments

We follow Schwarz's formulation [21]. The vortex line is represented by a space curve with its position described by  $\mathbf{S}(\xi, t)$ , where  $\xi$  parametrizes the curve and  $t$  is the time. The curl-free condition is violated on this curve,

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \kappa \int d\mathbf{S} \delta(\mathbf{r} - \mathbf{S}(\xi, t)), \quad (\text{A1})$$

where the integration is along the vortex line  $\mathbf{S}(\xi, t)$ . At distance relatively far from the vortex line, the above equation and the condition  $\nabla \cdot \mathbf{v} = 0$  yield a Biot-Savart type equation

$$\mathbf{v}_{\text{ind}} = \frac{\kappa}{4\pi} \int d\xi' \frac{[\mathbf{S}(\xi', t) - \mathbf{S}(\xi, t)] \times \mathbf{S}'_{\xi'}}{|\mathbf{S}(\xi', t) - \mathbf{S}(\xi, t)|^3} \quad (\text{A2})$$

where  $\mathbf{S}' = \partial \mathbf{S} / \partial \xi$ . The local superfluid velocity  $\mathbf{v}_{sl}$  can be affected by an external flow, more precisely  $\mathbf{v}_0$ , the superfluid velocity at large distance from any vortex line [11]

$$\mathbf{v}_{sl} = \mathbf{v}_{\text{ind}} + \mathbf{v}_0. \quad (\text{A3})$$

The Magnus force is given by

$$\mathbf{f}_M = \rho_s \kappa \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_L - \mathbf{v}_{sl}) \quad (\text{A4})$$

where  $\mathbf{v}_L = \dot{\mathbf{S}} \equiv d\mathbf{S}/dt$  is the velocity of the vortex line. The Magnus force reflects the difference between vortex line velocity,  $\mathbf{v}_L$ , and the local superfluid velocity,  $\mathbf{v}_{sl}$ . The next factor determining the vortex line dynamics is the mutual friction between the quantum vortices and the normal component of the superfluid. The mutual friction  $\mathbf{f}_D$  acting on a unit length of the vortex line is

$$\mathbf{f}_D = D_1 \frac{\mathbf{S}'}{|\mathbf{S}'|} \times \left[ \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}'_n - \dot{\mathbf{S}}) \right] + D_2 \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}'_n - \dot{\mathbf{S}}) \quad (\text{A5})$$

The vortex line motion is described by Schwarz's equation [21]

$$\dot{\mathbf{S}} = \mathbf{v}_{\text{ind}} + \mathbf{v}_0 + \alpha \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_{ns} - \mathbf{v}_{\text{ind}}) - \alpha' \frac{\mathbf{S}'}{|\mathbf{S}'|} \times \left[ \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_{ns} - \mathbf{v}_{\text{ind}}) \right] \quad (\text{A6})$$

where  $\mathbf{v}_{ns} \equiv \mathbf{v}_n - \mathbf{v}_s$  is difference between the average normal-fluid velocity and the applied superflow field, and the coefficients  $\alpha, \alpha'$  can be expressed in terms of the coefficients  $D_1, D_2$ . This is the basic equation for describing problems on the motion of the vortex lines, in particular, the *vortex tangle* problem [21].

We summarize the notations for vortex dynamics as follows,

- $\mathbf{S}(\xi, t)$  — position of the vortex line. ( $\dot{\mathbf{S}} \equiv d\mathbf{S}/dt, \mathbf{S}' \equiv \partial \mathbf{S} / \partial \xi, \dots$ );
- $\mathbf{v}_n$  — effective, or macroscopically averaged normal fluid velocity [21];
- $\mathbf{v}_s$  — macroscopically averaged superfluid velocity;
- $\mathbf{v}_L$  — vortex line velocity,  $\mathbf{v}_L = \dot{\mathbf{S}}$ ;

- $\mathbf{v}_{\text{ind}}$  — the velocity of superflow induced by the curvature of the vortex line;
- $\mathbf{v}_{sl}$  — local superfluid velocity,  $\mathbf{v}_{sl} = \mathbf{v}_{\text{ind}} + \mathbf{v}_s$ ;
- $\mathbf{f}_M$  — the Magnus force,  $\mathbf{f}_M = \rho_s \kappa \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_L - \mathbf{v}_{sl})$ ;
- $\mathbf{f}_D$  — the mutual friction has different expressions, e.g. [11, 21]  

$$\mathbf{f}_D = -\alpha \rho_s \kappa \frac{\mathbf{S}'}{|\mathbf{S}'|} \times \left[ \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_n - \mathbf{v}_{sl}) \right] - \alpha' \rho_s \kappa \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_n - \mathbf{v}_{sl})$$

- 
- [1] C. Xiong, M. Good, Y. Guo, X. Liu and K. Huang, Phys. Rev. D **90**, 125019 (2014), [arXiv:1408.0779 [hep-th]].
  - [2] L. Tisza, Nature, **141**, 913 (1938).
  - [3] L.D. Landau, Sov. Phys. JETP **11**, 592 (1941).
  - [4] P.L. Kapitsa, Nature, **141**, 74 (1932).
  - [5] I.M. Khalatnikov, *An introduction to the theory of superfluidity* (Benjamin, New York, 1965).
  - [6] R.N. Hill and P.H. Roberts, Int. J. Eng. Sci. **15**, 305 (1977); J. Low Temp. Phys. **30**, 709 (1978); J. Phys. C **11**, 4485 (1978).
  - [7] J.A. Geurst, Phys. Rev. B **22**, 307 (1980).
  - [8] V.V. Lebedev and I.M. Khalatnikov, Sov. Phys. JETP **56**, 923 (1982).
  - [9] E. B. Sonin, Rev. Mod. Phys. **59**, 87 (1987).
  - [10] R. P. Feynman, in *Progress in Low Temperature Physics, Vol. I*, ed. C. J. Gorter (North-Holland, Amsterdam, 1955), p17.
  - [11] R. J. Donnelly, *Quantized Vortices in Helium II*, Cambridge University Press, 1991.
  - [12] B. Carter and I. M. Khalatnikov, Phys. Rev. D **45**, 4536 (1992).
  - [13] B. Carter and D. Langlois, Phys. Rev. D **51**, 5855 (1995).
  - [14] P. Shygoiu and A. Svidzynski, Cond. Mat. Phys. **12**, 57 (2009).
  - [15] C. Coste, Eur. Phys. J. **B1**, 245 (1998).
  - [16] M. Good, C. Xiong, A. Chua and K. Huang, New J. Phys. **18**, 113018 (2016), [arXiv:1407.5760 [gr-qc]].
  - [17] M.G. Alford, S.K. Mallavarapu, A. Schmidt, and S. Stetina, Phys. Rev. D **87**, 065001 (2013).
  - [18] A. Vilenkin and E. P. S. Shellard, *Cosmic strings and other topological defects*, Cambridge University Press, 1994.
  - [19] B. Gradwohl, G Kalbermann, T. Piran, and E. Bertschinger, Nucl. Phys. B **338**, 371 (1990).
  - [20] M. Kalb and P. Ramond, Phys. Rev. D **9**, 2273 (1974).
  - [21] K. W. Schwarz, Phys. Rev. B **18**, 245 (1978); Phys. Rev. B **38**, 2398 (1998).
  - [22] K. Kawasaki, Physica **119A** (1983) 17.
  - [23] M. Tsubota, M. Kobayashi, H. Takeuchi, Phys. Rept. **523**, 191 (2013).
  - [24] S. K. Nemirovskii, Phys. Rep. **524**, (2013) 85.